Journal of Combinatorial Theory, Series A 117 (2010) 981–995

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

On IP-graphs of association schemes and applications to group theory

Bangteng Xu

Department of Mathematics and Statistics, Eastern Kentucky University, Richmond, KY 40475, United States

A R T I C L E I N F O

Article history:
Received 10 February 2009
Available online 22 August 2009

Keywords:
Association schemes
Closed subsets
Quotient schemes
IP-graphs
Schurian schemes
Common divisor graphs

A B S T R A C T

Let G be a group acting transitively on a set X such that all subdegrees are finite. Isaacs and Praeger (1993) [5] studied the common divisor graph of (G, X). For a group G and its subgroup A, based on the results in Isaacs and Praeger (1993) [5], Kaplan (1997) [6] proved that if A is stable in G and the common divisor graph of (A, G) has two components, then G has a nice structure. Motivated by the notion of the common divisor graph of (G, X), Camina (2008) [3] introduced the concept of the IP-graph of a naturally valenced association scheme. The common divisor graph of (G, X) is the IP-graph of the association scheme arising from the action of G on X. Xu (2009) [8] studied the properties of the IP-graph of an arbitrary naturally valenced association scheme, and generalized the main results in Isaacs and Praeger (1993) [5] and Camina (2008) [3]. In this paper we first prove that if the IP-graph of a naturally valenced association scheme (X, S) is stable and has two components (not including the trivial component whose only vertex is 1), then S has a closed subset T such that the thin residue $O^{\vartheta}(T)$ and the quotient scheme $(X/O^{\vartheta}(T), S//O^{\vartheta}(T))$ have very nice properties. Then for an association scheme (X, S) and a closed subset T of S such that $S//T$ is an association scheme on X/T, we study the relations between the closed subsets of S and those of $S//T$. Applying these results to schurian schemes and common divisor graphs of groups, we obtain the results of Kaplan [6] as direct consequences.

E-mail address: bangteng.xu@eku.edu.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2009.08.001

1. Introduction

Let G be a group acting transitively on a set X such that all subdegrees are finite. In order to study the relations among all subdegrees of (G, X), Isaacs and Praeger [5] introduced the concept of the common divisor graph of (G, X), and investigated the connectivity of the graph. The main results in [5] deal with the number of connected components of the graph, and the diameter of each non-trivial component. Note that the common divisor graph of (G, X) always has a component consisting of the single vertex 1. We call this component the trivial component, and any other components the non-trivial components. (It is possible that some non-trivial component may contain only one vertex.) Isaacs and Praeger [5] proved that the common divisor graph of (G, X) has at most two non-trivial connected components. If the common divisor graph of (G, X) has two non-trivial components, then Isaacs and Praeger [5] proved that one of these two components is a complete graph, and the other component has diameter at most two. The concept of paired subdegrees plays an important role in [5]. If the common divisor graph has two non-trivial components, then there are two cases to consider: any paired subdegrees lie in the same component or some paired subdegrees lie in different components. The properties of the components in each case were also given in [5].

Let G be a group, A a subgroup of G, and X the set of all left cosets of A in G. Then G acts on X transitively by left multiplication. If all subdegrees of (G, X) are finite, then the common divisor graph of (G, X) is called the common divisor graph of (A, G) by Kaplan [6]. Assume that the common divisor graph of (A, G) has two non-trivial components with vertex sets D_1 and D_2, respectively, such that D_1 contains the minimal element of $D \setminus \{1\}$, where D is the set of all subdegrees of (A, G). If any paired subdegrees lie in the same component, then by applying the results in [5], Kaplan [6] proved that the group G has a subgroup H such that $N_G(A) \subset H \subset G$, where $N_G(A)$ is the normalizer of A in G, and the set of all subdegrees of (A, H) is $D_1 \cup \{1\}$. Furthermore, let $L = A^H$, the normal closure of A in H. Then the index $|L : A|$ is finite and divides every element in D_2, and the set of all subdegrees of (L, G) is $\{m/|L : A| \mid m \in D_2\} \cup \{1\}$.

The common divisor graph of (G, X) is called the *IP graph* of (G, X) by Neumann [7]. Neumann [7] has introduced a variant of the IP graph, called the *VIP graph*. Other related research can be found in [1] and [2], etc.

Let G be a group acting transitively on a set X such that all subdegrees are finite. Then the action of G on X induces a naturally valenced association scheme S on X, and the subdegrees of (G, X) are the valencies of elements in S. Motivated by the concept of the common divisor graph of (G, X), Camina [3] introduced the IP-graph of a naturally valenced association scheme. The common divisor graph of (G, X) is the IP-graph of the naturally valenced association scheme (X, S) arising from the action of G on X. For a naturally valenced association scheme (X, S), its IP-graph provides a way to measure how far valencies of elements of S can be independent in a certain sense. Camina [3] proved that the main results of [5] are also true for the IP-graph of a naturally valenced association scheme that satisfies a very strong restriction: any paired valencies are equal. However, we note that the common divisor graph of (G, X) studied in [5] does not satisfy this restriction.

Xu [8] studied the IP-graph of an arbitrary naturally valenced association scheme, and proved that the main results of [5] are also true for the IP-graph of any naturally valenced association scheme, without the assumption that paired valencies are equal. In particular, Xu [8] proved that the IP-graph $\mathcal{IP}(S)$ of a naturally valenced association scheme (X, S) has at most two non-trivial components. (Note that the IP-graph $\mathcal{IP}(S)$ always has a component consisting of the single vertex 1. We call this component the trivial component, and any other components the non-trivial components. It is possible that some non-trivial component may contain only one vertex.) If the graph $\mathcal{IP}(S)$ has two non-trivial components, then Xu [8] proved that one of these two components is a complete graph, and the other component has diameter at most two. When the graph $\mathcal{IP}(S)$ has two non-trivial components, there are two cases to consider: the valencies n_s and n_{s^*} lie in the same component for any $s \in S$ or the valencies n_s and n_{s^*} lie in different components for some $s \in S$. Properties of the components in each case were also studied in [8].

In this paper, based on the results in [8], we study the algebraic structure of a naturally valenced association scheme (X, S) whose IP-graph has two non-trivial components such that the valencies n_s and n_{s^*} lie in the same component for any $s \in S$. We will prove that (X, S) has a closed subset T such that the thin residue $O^{\vartheta}(T)$ and the quotient scheme $(X/O^{\vartheta}(T), S//O^{\vartheta}(T))$ have very nice properties (Theorems 3.8 and 3.10). Then for an association scheme (X,S) and a closed subset T of S such that $S//T$ is an association scheme on X/T, we study the relations between the closed subsets of S and those of $S//T$. We will prove a correspondence theorem and generalize many known results. As an application, for a group G and its subgroup A, we will obtain the relationship between the closed subsets of the schurian scheme $S(A,G)$ and the subgroups of G containing A (Theorem 4.7). As direct consequences of Theorems 3.8, 3.10, and 4.7, we will obtain the results in [6] very easily.

The rest of the paper is organized as follows. We will start with necessary statements of definitions, examples, and some known results of association schemes and IP-graphs of naturally valenced association schemes in Section 2. Section 3 is devoted to study the structure of a naturally valenced association scheme (X,S) whose IP-graph has two non-trivial components such that the valencies n_s and n_{s^*} lie in the same component for any $s \in S$. For an association scheme (X,S) and a closed subset T of S such that $S//T$ is an association scheme on X/T, we study the relations between the closed subsets of S and those of $S//T$, and applications to schurian schemes in Section 4. Finally in Section 5, by applying the results proved in Sections 3 and 4, we obtain the results in [6] as direct consequences.

2. IP-graphs of association schemes

This section is devoted to statements of definitions, examples, and some known results of association schemes and IP-graphs of naturally valenced association schemes. Let us first state the definition of an association scheme. The following definition is adapted from the book of Zieschang [10]. Let X be a set, and S a partition of $X \times X$. Then S is called an *association scheme* on X if the following properties hold:

(i) $1_X \in S$, where $1_X := \{(x,x) \mid x \in X\}$. (Usually we simply denote 1_X by 1.)
(ii) For any $s \in S$, s^* is also in S, where $s^* := \{(y,z) \mid (z,y) \in s\}$.
(iii) For any $p,q,r \in S$, there exists a cardinal number a_{pqr} such that for any $(y,z) \in r$, the cardinality $|\{x \in X \mid (y,x) \in p \text{ and } (x,z) \in q\}| = a_{pqr}$. ($a_{pqr}$ are called the structure constants of S.)

Let (X,S) be an association scheme. For any $x \in X$ and $s \in S$, define $xs := \{y \in X \mid (x,y) \in s\}$. Then the equation in the above property (iii) can be written as $|yp \cap zq^*| = a_{pqr}$. For any $s \in S$, the *valency* of s is defined by $n_s := a_{ss^*1}$. Recall that for each $x \in X$, $n_s = |xs|$, the cardinality of the set xs. If for any $s \in S$, n_s is finite, then (X,S) is called a *naturally valenced* association scheme. There may exist $s \in S$ such that $n_s \neq n_{s^*}$. If $n_s = n_{s^*}$ for all $s \in S$, then we say that *paired valencies are equal* (see [3, Definition 3]). An element $s \in S$ is called a *thin* element of S if $n_s = 1$. Note that $n_1 = 1$, *i.e.* 1_X is a thin element of S. If $n_s = 1$ for all $s \in S$, then S is called a *thin scheme*. There may exist $s \in S$ such that s is a thin element but s^* is not thin. Such examples can be found in Section 5 of [5]. However, if X is a finite set, then for any $s \in S$, $n_s = n_{s^*}$, and hence s is thin if and only if s^* is also thin. But in this paper we do not assume that X is a finite set.

Now we introduce the definition of the IP-graph of a naturally valenced association scheme. The next definition is adapted from [3, Definition 2].

Definition 2.1. (See [3, Definition 2].) Let (X,S) be a naturally valenced association scheme. The *IP-graph* of S, denoted by $\mathcal{IP}(S)$, is the undirected graph with vertex set $\{n_s \mid s \in S\}$ such that two distinct vertices n_r and n_s are joined by an edge if n_r and n_s are not coprime.

One of the most important examples of IP-graphs is the common divisor graph of a group G acting transitively on a set X, studied by Isaacs and Praeger [5] and Kaplan [6]. We will need some concepts and notation in [5] and [6] in our discussions. The next two examples give the necessary details about the common divisor graphs. Example 2.2 is adapted from [5], and Example 2.3 is adapted from [6].

Example 2.2. Let G be a group acting transitively on a set X. Then G acts on $X \times X$ by $g(x,y) := (gx, gy)$, for any $g \in G$ and any $x, y \in X$. Let S be the set of all orbits of G on $X \times X$. Then (X,S)

is an association scheme, called the association scheme *arising from the action of G on X*. Fix $x \in X$ for the rest of this example. Let G_x be the stabilizer of x. Then the lengths (cardinalities) of orbits of G_x on X are called *subdegrees* of (G, X) (see [4, p. 13]). Note that the set of all orbits of G_x on X is $\{xs \mid s \in S\}$, and for any $s \in S$, the valency $n_s = |xs|$. So the set of valencies of elements in S and the set of subdegrees of (G, X) are the same. Note that 1 is always a subdegree of (G, X). Assume that the subdegrees of (G, X) are all finite. Then the *common divisor graph* of (G, X) introduced by Isaacs and Praeger [5] is the IP-graph of the association scheme (X, S) arising from the action of G on X. Let $A := G_x$. Then for any $g \in G$, the length of the A-orbit of the element $gx \in X$ is equal to the index $|A : A \cap gAg^{-1}|$. Thus,

$$\{n_s \mid s \in S\} = \{|A : A \cap gAg^{-1}| \mid g \in G\}.$$

Note that for any $g \in G$, there is $s \in S$ such that $gx \in xs$. Hence $g^{-1}x \in xs^*$. Thus, $|xs|$ is equal to the length of the A-orbit of gx, and $|xs^*|$ is equal to the length of the A-orbit of $g^{-1}x$. Therefore,

$$n_s = |A : A \cap gAg^{-1}| \quad \text{and} \quad n_{s^*} = |A : A \cap g^{-1}Ag| = |gAg^{-1} : gAg^{-1} \cap A|.$$

Two subdegrees m and m^* of (G, X) are called *paired* by Isaacs and Praeger [5] if there exists $g \in G$ such that $m = |A : A \cap gAg^{-1}|$ and $m^* = |gAg^{-1} : gAg^{-1} \cap A|$. So subdegrees m and m^* of (G, X) are paired if and only if there exists $s \in S$ such that $m = n_s$ and $m^* = n_{s^*}$. The concept of paired subdegrees plays an important role in [5]. Clearly this concept is very natural in the context of association schemes. Note that there may exist $s, t \in S$ such that $n_s = n_t$ but $n_{s^*} \neq n_{t^*}$. (Such examples can be found in [5, Section 5].) So a subdegree may be paired with more than one subdegree, and every subdegree is paired with some other subdegree or with itself.

Remark. Let Z be a set of positive integers. Then we can define the *common divisor graph $\Gamma(Z)$ on Z* to be the undirected graph whose vertex set is Z such that any two distinct vertices m and n are joined by an edge if they are not coprime. So the IP-graph of a naturally valenced association scheme (X, S) is the common divisor graph on the set of valencies of elements in S.

Example 2.3. Let G be a group, and A a subgroup of G. Let X be the set of all left cosets of A in G, *i.e.* $X := \{gA \mid g \in G\}$. Then G acts on X by left multiplication; that is, for any $g \in G$ and any $uA \in X$, $g \cdot uA := guA$. As in the above example, G acts on $X \times X$ by $g(uA, vA) := (guA, gvA)$, for any $g \in G$ and any $uA, vA \in X$. Let $S(A, G)$ be the set of all orbits of G on $X \times X$. Since the action of G on X is transitive, $(X, S(A, G))$ is an association scheme, called the *schurian scheme* of (A, G). Let $u \in G$. Then there exists $s \in S(A, G)$ such that $(A, uA) \in s$, and hence $s = \{g(A, uA) \mid g \in G\}$. Thus, the valency n_s is equal to the cardinality of the set $\{vA \mid v \in G \text{ and } vA \subseteq AuA\}$. As in [6], this cardinality is called a *subdegree* of (A, G), and denoted by $[AuA : A]$. Note that for any $s \in S(A, G)$, if $(A, uA) \in s$ for some $u \in N_G(A)$, the normalizer of A in G, then s is a thin element. Hence, $S(A, G)$ is a thin scheme if and only if A is normal in G. Assume that A is not normal in G, and all the subdegrees of (A, G) are finite. Then the IP-graph $\mathcal{IP}(S(A, G))$ is called the *common divisor graph* of (A, G). Note that

$$\{n_s \mid s \in S(A, G)\} = \{[AgA : A] \mid g \in G\}.$$

Furthermore, for any $g \in G$, if $(A, gA) \in s$ for some $s \in S(A, G)$, then $(A, g^{-1}A) \in s^*$, and hence

$$n_s = [AgA : A] \quad \text{and} \quad n_{s^*} = [Ag^{-1}A : A].$$

As in [6], two subdegrees m and m^* of (A, G) are called *paired* if there exists $g \in G$ such that $m = [AgA : A]$ and $m^* = [Ag^{-1}A : A]$. So m and m^* are paired if and only if there exists $s \in S(A, G)$ such that $m = n_s$ and $m^* = n_{s^*}$. Note that Kaplan [6] used right cosets instead of left cosets of A in G; but essentially there is no difference for the discussions of the common divisor graph. Let $\mathcal{IP}(S(A, G)) - \{1\}$ be the subgraph of $\mathcal{IP}(S(A, G))$ obtained by deleting the vertex 1. Then $\mathcal{IP}(S(A, G)) - \{1\}$ is called the common divisor graph of (A, G) by Kaplan [6]. To be consistent with the definitions in [5,3,8], in this paper the common divisor graph of (A, G) will be $\mathcal{IP}(S(A, G))$ instead of $\mathcal{IP}(S(A, G)) - \{1\}$. Furthermore, the subgroup A is called *stable* in G by Kaplan [6] if any paired subdegrees of (A, G) lie in the same connected component of the common divisor graph of (A, G).

Remark. Different formulas for subdegrees and paired subdegrees are used in Examples 2.2 and 2.3 by Isaacs, Praeger, and Kaplan. However, these formulas are equivalent. In Example 2.3, if we take $x = A \in X$, then the stabilizer $G_x = A$, $[AgA : A] = |A : A \cap gAg^{-1}|$, and $[Ag^{-1}A : A] = |A : A \cap g^{-1}Ag| = |gAg^{-1} : gAg^{-1} \cap A|$. So these formulas are indeed equivalent.

Properties of IP-graphs of naturally valenced association schemes have been studied in [8]. One of the main results in [8] is the following theorem. Recall that the IP-graph $\mathcal{IP}(S)$ of a naturally valenced association scheme (X, S) always has a component consisting of the single vertex 1. We call this component the trivial component, and any other components the non-trivial components. It is possible that some non-trivial component may contain only one vertex.

Theorem 2.4. *(See [8, Theorem 1.1].) Let (X, S) be a naturally valenced association scheme. Then the graph $\mathcal{IP}(S)$ has at most two non-trivial components.*

Let (X, S) be a naturally valenced association scheme. Similar to the definition in [6], the graph $\mathcal{IP}(S)$ is called *stable* if for any $s \in S$, n_s and n_{s^*} lie in the same component. In this paper our interest is focused on the naturally valenced association scheme (X, S) whose IP-graph $\mathcal{IP}(S)$ is stable and has two non-trivial components. Such examples can be found in both [5] and [6]. For such an IP-graph, the following theorem describes the properties of the non-trivial components. Our discussions in this paper are based on this theorem. A similar result for the common divisor graph of a group G acting transitively on a set X can be found in [5, (4.2) Theorem]. Also see [6, Theorem E].

Let (X, S) be a naturally valenced association scheme. Let $p \in S$. Then the component of the graph $\mathcal{IP}(S)$ that has the vertex n_p is denoted by $\mathcal{C}(n_p)$. If $n_p > 1$ and $n_p \leqslant n_s$ for any $s \in S$ such that $n_s \neq 1$, then n_p is called a *minimal vertex* of $\mathcal{IP}(S)$.

Theorem 2.5. *(See [8, Theorem 5.3].) Let (X, S) be a naturally valenced association scheme such that the graph $\mathcal{IP}(S)$ is stable and has two non-trivial components. Let $p \in S$ such that n_p is the minimal vertex of the graph $\mathcal{IP}(S)$, and let $q \in S$ such that $n_q > 1$ is not a vertex of the component $\mathcal{C}(n_p)$. Then the following hold.*

(i) *The component $\mathcal{C}(n_q)$ is a complete graph.*
(ii) *For any $r \in S$ such that n_r is a vertex of the component $\mathcal{C}(n_p)$, we have that $n_r = n_{r^*}$, and n_r is less than the greatest common divisor of the vertices (integers) in the component $\mathcal{C}(n_q)$.*
(iii) *The component $\mathcal{C}(n_p)$ has a maximal vertex n_t, i.e. $n_t > n_s$ for any other vertex n_s of $\mathcal{C}(n_p)$. Furthermore, the maximal vertex n_t is adjacent to any other vertex of $\mathcal{C}(n_p)$. In particular, the component $\mathcal{C}(n_p)$ is a finite graph and has diameter at most two.*

3. Closed subsets and quotient schemes arising from IP-graphs

In this section we study the naturally valenced association scheme (X, S) whose IP-graph is stable and has two non-trivial components. We will prove that (X, S) has a closed subset T such that the thin residue $O^\vartheta(T)$ and the quotient scheme $(X/O^\vartheta(T), S//O^\vartheta(T))$ have very nice properties.

Let (X, S) be an association scheme. Then for any $p, q \in S$, define $pq := \{r \in S \mid a_{pqr} \neq 0\}$, and for any nonempty subsets P and Q of S, define $PQ := \{r \in S \mid$ there exists $p \in P$ and $q \in Q$ such that $a_{pqr} \neq 0\}$. Thus, for any $r \in S$, $r \in PQ$ if and only if $r \in pq$ for some $p \in P$ and $q \in Q$. Note that for any nonempty subsets P, Q, and R of S, $(PQ)R = P(QR)$ by [10, Lemma 1.3.1]; so we will write both $(PQ)R$ and $P(QR)$ as PQR. It is clear that for any $s \in S$, $s \in PQR$ if and only if $s \in pqr$ for some $p \in P$, $q \in Q$, and $r \in R$. For any $s \in S$ and any nonempty subsets P and Q of S, we will write $\{s\}P$ as sP, $P\{s\}$ as Ps, and $P\{s\}Q$ as PsQ, etc.

Let (X, S) be an association scheme, and T a nonempty subset of S. Define $T^* = \{t^* \mid t \in T\}$. Then T is called a *closed subset* of S if $T^*T \subseteq T$. If T is a closed subset of S, then $1 \in T$, $T^* = T$, and $TT = T$. Clearly the intersection of closed subsets of S is again a closed subset of S. Let P be a nonempty subset of S. Then the intersection of all closed subsets of S that contain P is called the closed subset of S *generated* by P, and denoted by $\langle P\rangle$. (P is called a *set of generators* of $\langle P\rangle$.) From [10, Lemma 3.1.1(i)], $\langle P\rangle$ is the union of the sets $(P^*\cup P)^n$ for all non-negative integers n, where $(P^*\cup P)^0 := \{1\}$, and for each positive integer n, $(P^*\cup P)^n$ is inductively defined by $(P^*\cup P)^n := (P^*\cup P)^{n-1}(P^*\cup P)$. Since $1\in(P^*\cup P)^2$, we see that

$$\langle P\rangle = \bigcup_{n\in\mathbb{N}} \left(P^*\cup P\right)^n, \tag{3.1}$$

where $\mathbb{N}$ is the set of all positive integers.

Let (X,S) be an association scheme, and T a nonempty subset of S. Then the *valency* of T, n_T, is defined by

$$n_T := \sum_{s\in T} n_s.$$

Properties of valencies of subsets of an association scheme are studied in [10] and [8]. The next lemma is important in our discussions.

Lemma 3.1. *(See [8, Proposition 2.6].) Let (X,S) be an association scheme. Let P and Q be nonempty subsets of S such that $1\in P$ and $P^*=P$. Then the following hold.*

(i) *If $PQ=Q$ and $n_{Q^*}<\infty$, then $n_{\langle P\rangle}<\infty$, and $n_{\langle P\rangle}\mid n_{Q^*}$.*
(ii) *If $QP=Q$ and $n_Q<\infty$, then $n_{\langle P\rangle}<\infty$, and $n_{\langle P\rangle}\mid n_Q$.*

Let (X,S) be a naturally valenced association scheme, and $p,q\in S$. Then the distance between vertices n_p and n_q of the graph $\mathcal{IP}(S)$ is denoted by $d(n_p,n_q)$. Thus, $d(n_p,n_q)=0$ if $n_p=n_q$, and $d(n_p,n_q)=\infty$ if n_p and n_q are vertices of different components. Furthermore, the greatest common divisor of the two integers n_p and n_q is denoted by (n_p,n_q). So for any two distinct vertices n_p and n_q, $d(n_p,n_q)\geqslant 2$ if and only if $(n_p,n_q)=1$. For any two nonzero integers m and n, if m divides n, then we denote $m\mid n$.

The next lemma plays an important role in our discussions. The reader is referred to [8] for more details.

Lemma 3.2. *(See [8, Lemma 3.2(i), (ii)].) Let (X,S) be a naturally valenced association scheme. Let $p,q\in S$. Then the following hold.*

(i) *If $d(n_p,n_q)\geqslant 3$, $(n_{p^*},n_q)=1$, and $n_p<n_q$, then pq contains exactly one element of S, and*

$$n_{pq}=n_q,\quad p^*pq=\{q\},\quad n_{\langle p^*p\rangle}\mid n_{q^*}\quad \textit{and}\quad n_{(pq)^*}=\frac{n_{p^*}n_{q^*}}{n_p}.$$

(ii) *If $d(n_p,n_q)\geqslant 3$, $(n_{p^*},n_q)=1$, and $n_p>n_q$, then pq contains exactly one element of S, and $n_{pq}=n_p$. Furthermore, if $n_q=n_{q^*}$, then*

$$pqq^*=\{p\},\quad pq^*q=\{p\},\quad n_{\langle qq^*\rangle}\mid n_p,\quad n_{\langle q^*q\rangle}\mid n_p,\quad \textit{and}\quad n_{p^*}=n_{(pq)^*}=n_{qp^*}.$$

Remark. Lemma 3.2(ii) cannot be obtained from Lemma 3.2(i) by switching p and q. For any two nonempty subsets P and Q of S, we have that $n_Q\leqslant n_{PQ}$, and $n_Q=n_{PQ}$ if and only if $Q=P^*PQ$ by [10, Lemma 1.4.4]. However, we do not have $n_P\leqslant n_{PQ}$ in general. What we have are $n_P\leqslant n_{PQ}n_{Q^*}/n_Q$, and if $n_{Q^*}<\infty$, then $n_P=n_{PQ}n_{Q^*}/n_Q$ if and only if $P=PQQ^*$ by [8, Proposition 2.2(ii)].

Let (X,S) be a naturally valenced association scheme. Thin elements of S are important in the study of the IP-graph $\mathcal{IP}(S)$. Note that S may have a thin element s such that s^* is not thin. The next lemma is part of [8, Theorem 1.2]. Recall that for any $s\in S$, the component of $\mathcal{IP}(S)$ that has the vertex n_s is denoted by $\mathcal{C}(n_s)$.

Lemma 3.3. *Let (X, S) be a naturally valenced association scheme. If S has a thin element s such that s^* is not thin, then the IP-graph $\mathcal{IP}(S)$ has only one non-trivial component.*

Let (X, S) be an association scheme. A closed subset T of S is called a *normal closed subset* if for any $s \in S$, $sT = Ts$. If T is a closed subset of S, and $s^*Ts = T$ for any $s \in S$, then T is called a *strongly normal closed subset*. Note that a strongly normal closed subset is also a normal closed subset by [10, Lemma 2.5.5], and the intersection of strongly normal closed subsets is also a strongly normal closed subset.

Let (X, S) be an association scheme, and T a closed subset of S. Let R be a nonempty subset of T. Then the intersection of all strongly normal closed subsets of T that contain R is called the *strongly normal closure* of R in T, and denoted by $\mathrm{snc}(R, T)$. Note that $\mathrm{snc}(R, T)$ is a strongly normal closed subset of T that contains R. The next lemma gives a set of generators of $\mathrm{snc}(R, T)$.

Lemma 3.4. *Let (X, S) be an association scheme, and T a closed subset of S. Let R be a nonempty subset of T. Then*

$$\mathrm{snc}(R, T) = \Big\langle \bigcup_{t\in T,\, r\in R} t^*rt \Big\rangle.$$

Proof. Let $U = \bigcup_{t\in T,\, r\in R} t^*rt$. Then U and hence $\langle U\rangle$ are contained in any strongly normal closed subset of T that contains R. Thus, $\langle U\rangle \subseteq \mathrm{snc}(R, T)$. In the following we show that $\langle U\rangle$ is a strongly normal closed subset of T. Let $p, q \in T$ and $r \in R$. Then $(q^*rq)(q^*r^*q) \subseteq \langle U\rangle$. Hence, $q^*q \subseteq q^*[(rq)(rq)^*]q$ implies that $q^*q \subseteq \langle U\rangle$. Thus, for any $s \in pq$, we see that $p \in sq^*$, and hence $s^*rpq \subseteq (s^*rs)q^*q \subseteq \langle U\rangle$. Therefore, $q^*(p^*rp)q \subseteq \langle U\rangle$ for any $p, q \in T$ and $r \in R$. So for any $t \in T$, $t^*(U \cup U^*)t \subseteq \langle U\rangle$, and hence for any positive integer n, $t^*(U \cup U^*)^n t \subseteq [t^*(U \cup U^*)t]^n \subseteq \langle U\rangle$. Thus, (3.1) implies that for any $t \in T$, $t^*\langle U\rangle t \subseteq \langle U\rangle$. So $\langle U\rangle$ is a strongly normal closed subset of T. But $U \supseteq R$. Hence, $\langle U\rangle \supseteq \mathrm{snc}(R, T)$. Therefore, $\langle U\rangle = \mathrm{snc}(R, T)$, and Lemma 3.4 holds. □

Let (X, S) be an association scheme, and T a closed subset of S. The intersection of all strongly normal closed subsets of T is called the *thin residue* of T, and denoted by $O^\vartheta(T)$. Note that $O^\vartheta(T)$ is a strongly normal closed subset of T, and $O^\vartheta(T) = \mathrm{snc}(\{1\}, T)$. So we have the following corollary.

Corollary 3.5. *(See [10, Theorem 3.2.1(ii)].) Let (X, S) be an association scheme, and T a closed subset of S. Then*

$$O^\vartheta(T) := \Big\langle \bigcup_{t\in T} t^*t \Big\rangle.$$

Now we are ready to prove some structural results for a naturally valenced association scheme whose IP-graph $\mathcal{IP}(S)$ is stable and has two non-trivial components.

Lemma 3.6. *Let (X, S) be a naturally valenced association scheme such that the graph $\mathcal{IP}(S)$ has two non-trivial components. Let $p \in S$ such that n_p is the minimal vertex of the graph $\mathcal{IP}(S)$, and $T := \{s \in S \mid n_s = 1$ or n_s is a vertex of the component $\mathcal{C}(n_p)\}$. Then T is a closed subset of S if and only if $\mathcal{IP}(S)$ is stable.*

Proof. If T is a closed subset of S, then for any $s \in T$, we have $s^* \in T$. Since $\mathcal{IP}(S)$ has two non-trivial components, for any thin element $s \in S$, s^* is also thin by Lemma 3.3. Thus, for any $s \in S$ such that n_s is a vertex of the component $\mathcal{C}(n_p)$, n_{s^*} is also a vertex of $\mathcal{C}(n_p)$. Hence, $\mathcal{IP}(S)$ is stable.

Now assume that $\mathcal{IP}(S)$ is stable. Let $s \in T$. Then $s^* \in T$. Thus, to prove that T is a closed subset of S, it is enough to show that for any $s, t \in T$, $st \subseteq T$. Let $s, t \in T$ and $r \in st$. Then we prove that $r \in T$. Toward a contradiction, suppose that $r \notin T$. Then $s, s^* \in T$ implies that $d(n_{s^*}, n_r) = \infty$, $(n_s, n_r) = 1$, and $n_{s^*} < n_r$ by Theorem 2.5(ii). Thus, from Lemma 3.2 we get that $s^*r = \{q\}$ for some $q \in S$, and $n_{s^*r} = n_r$. So $n_q = n_r$, and hence $q \notin T$. But $r \in st$ implies that $t \in s^*r$. So $t = q \notin T$, a contradiction. This proves that $r \in T$, and hence $st \subseteq T$ for any $s, t \in T$. So T is a closed subset of S, and the lemma holds. □

Hypothesis 3.7. Let (X, S) be a naturally valenced association scheme. Assume that the graph $\mathcal{IP}(S)$ is stable and has two non-trivial components, and $T := \{s \in S \mid n_s = 1$ or n_s is a vertex of the component $\mathcal{C}(n_p)\}$, where $p \in S$ such that n_p is the minimal vertex of the graph $\mathcal{IP}(S)$. Then T is a closed subset of S by Lemma 3.6.

Theorem 3.8. *Let (X, S) be a naturally valenced association scheme that satisfies Hypothesis* 3.7. *Then the following hold.*

(i) *The thin residue $O^{\vartheta}(T)$ is a finite normal closed subset of S, and for any $s \in S \setminus T$,*

$$O^{\vartheta}(T)s = sO^{\vartheta}(T) = \{s\} \quad \textit{and} \quad n_{O^{\vartheta}(T)} \mid n_s.$$

(ii) *For any closed subset U of S, either $U \subseteq T$ or $U \supseteq O^{\vartheta}(T)$.*

Proof. (i) Let $s \in S \setminus T$. Then for any $t \in T$, as in the proof of Lemma 3.6, we have that $d(n_t, n_s) = \infty$, $(n_{t^*}, n_s) = 1$, and $n_t < n_s$. Thus, by Lemma 3.2 we get that $t^*ts = \{s\}$. Let $P = \bigcup_{t\in T} t^*t$. Then $Ps = \{s\}$. Hence, $P^n s = \{s\}$ for any positive integer n. Since $1 \in P$ and $P^* = P$, we see that $\langle P\rangle = \bigcup_{n\in\mathbb{N}} P^n$ by (3.1). Thus, $\langle P\rangle s = \{s\}$ for any $s \in S \setminus T$. But the thin residue $O^{\vartheta}(T) = \langle P\rangle$ by Corollary 3.5. Hence, $O^{\vartheta}(T)s = \{s\}$ for any $s \in S \setminus T$. Furthermore, since T is a closed subset of S, $s \in S \setminus T$ implies that $s^* \in S \setminus T$. Thus, from what we have just proved, we also have that $O^{\vartheta}(T)s^* = \{s^*\}$. Since $[O^{\vartheta}(T)]^* = O^{\vartheta}(T)$, from [10, Lemma 1.3.2(iii)] we see that $\{s\} = [O^{\vartheta}(T)s^*]^* = s[O^{\vartheta}(T)]^* = sO^{\vartheta}(T)$. That is, $sO^{\vartheta}(T) = \{s\}$ for any $s \in S \setminus T$. Thus, we have proved that

$$O^{\vartheta}(T)s = sO^{\vartheta}(T) = \{s\}, \quad \text{for any } s \in S \setminus T. \tag{3.2}$$

Since $O^{\vartheta}(T)$ is a strongly normal closed subset of T, (3.2) implies that $O^{\vartheta}(T)$ is a normal closed subset of S. Furthermore, for any $s \in S \setminus T$, since $O^{\vartheta}(T)s^* = \{s^*\}$ and $n_s < \infty$, we see that $n_{O^{\vartheta}(T)} < \infty$ and $n_{O^{\vartheta}(T)} \mid n_s$ by Lemma 3.1. In particular, $O^{\vartheta}(T)$ is a finite subset of S. So (i) holds.

(ii) Let $s \in S \setminus T$. Then $O^{\vartheta}(T)s = \{s\}$ by (i). Thus, for any $r \in O^{\vartheta}(T)$, we have $s \in rs$, and hence $r \in ss^*$. So $O^{\vartheta}(T) \subseteq ss^*$ for any $s \in S \setminus T$. Let U be a closed subset of S such that U is not contained in T. Then there exists $s \in U$ such that $s \in S \setminus T$. Hence, $U \supseteq ss^* \supseteq O^{\vartheta}(T)$. So (ii) holds. □

Let (X, S) be an association scheme. Let P be a nonempty subset of S. Then for any $x \in X$, define

$$xP := \{y \in X \mid (x, y) \in s \text{ for some } s \in P\}.$$

Corollary 3.9. *Let (X, S) be a naturally valenced association scheme that satisfies Hypothesis* 3.7. *Then for any $s \in S \setminus T$, the following hold.*

(i) *Let $r \in TsT$. Then $n_r = n_s$.*
(ii) *Let $x, y \in X$ such that $(x, y) \in s$. Then $xO^{\vartheta}(T) \times yO^{\vartheta}(T) \subseteq s$.*

Proof. (i) Let $t_1 \in T$. Then $t_1^* \in T$. Hence, for any $s \in S \setminus T$, $d(n_{t_1}, n_s) = \infty$, $(n_{t_1^*}, n_s) = 1$, and $n_{t_1} < n_s$ by Theorem 2.5(ii). Thus, Lemma 3.2 yields that $t_1 s$ contains exactly one element of S, and $n_{t_1 s} = n_s$. Let $t_1 s = \{q\}$. Then $n_q = n_s$. Thus, $q \in S \setminus T$, and hence $q^* \in S \setminus T$. So for any $t_2 \in T$, $d(n_q, n_{t_2}) = \infty$, $(n_{q^*}, n_{t_2}) = 1$, and $n_q > n_{t_2}$ by Theorem 2.5(ii). Therefore, Lemma 3.2 implies that qt_2 contains exactly one element of S, and $n_{qt_2} = n_q$. Thus, we have proved that for any $t_1, t_2 \in T$ and $s \in S \setminus T$, $t_1 s t_2$ contains exactly one element of S, and $n_{t_1 s t_2} = n_s$. Hence, for any $s \in S \setminus T$ and any $r \in TsT$, since $r \in t_1 s t_2$ for some $t_1, t_2 \in T$, we see that $t_1 s t_2 = \{r\}$, and hence $n_r = n_s$. So (i) holds.

(ii) Let $s \in S \setminus T$ and $x, y \in X$ such that $(x, y) \in s$. Then for any $x_1 \in xO^{\vartheta}(T)$ and $y_1 \in yO^{\vartheta}(T)$, there are $u, v \in O^{\vartheta}(T)$ such that $(x, x_1) \in u$ and $(y, y_1) \in v$. Thus, $(x_1, x) \in u^*$, $(x, y) \in s$, and $(y, y_1) \in v$ imply that $(x_1, y_1) \in r$ for some $r \in u^*sv$. But $O^{\vartheta}(T)sO^{\vartheta}(T) = \{s\}$ by Theorem 3.8(i). So $u^*sv = \{s\}$, and hence $(x_1, y_1) \in s$. This proves that $xO^{\vartheta}(T) \times yO^{\vartheta}(T) \subseteq s$, and (ii) holds. □

Let (X, S) be an association scheme, and T a closed subset of S. Then $\{xT \mid x \in X\}$ is a partition of X (see [10, Lemma 2.1.4(b)]). Let $X/T := \{xT \mid x \in X\}$. For any $s \in S$, define $s^T := \{(yT, zT) \mid z \in yTsT\}$, and define $S//T := \{s^T \mid s \in S\}$. Thus, for any $s \in S$ and any $y, z \in X$, $(yT, zT) \in s^T$ if and only if $(y, z) \in TsT$. Note that for any $p, q \in S$,

$$p^T = q^T \quad \Leftrightarrow \quad TpT = TqT \quad \Leftrightarrow \quad p^T \cap q^T \neq \emptyset \quad \Leftrightarrow \quad TpT \cap TqT \neq \emptyset$$

by [10, Lemma 4.1.1]. If S is naturally valenced and T is a finite closed subset of S, then by [10, Theorem 4.1.3], $S//T$ is an association scheme on X/T, and for any $p, q, r \in S$, the structure constant $a_{p^T q^T r^T}$ of $S//T$ is determined by

$$a_{p^T q^T r^T} = \frac{1}{n_T} \sum_{u \in TpT} \sum_{v \in TqT} a_{uvr}. \tag{3.3}$$

In particular, (3.3) implies that

$$n_{p^T} = \frac{n_{TpT}}{n_T}, \quad \text{for any } p \in S. \tag{3.4}$$

The association scheme $(X/T, S//T)$ is called the *quotient scheme of S over T* (see [10, p. 65]).

Theorem 3.10. *Let (X, S) be a naturally valenced association scheme that satisfies Hypothesis* 3.7. *Then the quotient scheme $(X/O^\vartheta(T), S//O^\vartheta(T))$ is defined, and for any $s \in S$, the element $s^{O^\vartheta(T)}$ of the quotient scheme $S//O^\vartheta(T)$ has valency*

$$n_{s^{O^\vartheta(T)}} = \begin{cases} 1, & \text{if } s \in T, \\ \frac{n_s}{n_{O^\vartheta(T)}}, & \text{if } s \notin T. \end{cases}$$

Proof. Since $O^\vartheta(T)$ is a finite closed subset of S by Theorem 3.8(i), we have the quotient scheme $(X/O^\vartheta(T), S//O^\vartheta(T))$ by [10, Theorem 4.1.3(i)]. Furthermore, for any $s \in S$, from (3.4) we get that

$$n_{s^{O^\vartheta(T)}} = \frac{n_{O^\vartheta(T)sO^\vartheta(T)}}{n_{O^\vartheta(T)}}.$$

If $s \in T$, since $O^\vartheta(T)$ is strongly normal in T, we see that $O^\vartheta(T)sO^\vartheta(T) = sO^\vartheta(T)$, and hence $n_{O^\vartheta(T)sO^\vartheta(T)} = n_{sO^\vartheta(T)}$. But $s^*sO^\vartheta(T) = O^\vartheta(T)$ for any $s \in T$ by Corollary 3.5. Thus, $n_{sO^\vartheta(T)} = n_{O^\vartheta(T)}$ by [10, Lemma 1.4.4(ii)]. Therefore, $n_{O^\vartheta(T)sO^\vartheta(T)} = n_{O^\vartheta(T)}$ for any $s \in T$. Hence, $n_{s^{O^\vartheta(T)}} = 1$ if $s \in T$. If $s \in S \setminus T$, then $O^\vartheta(T)sO^\vartheta(T) = \{s\}$ by Theorem 3.8(i). Thus, $n_{s^{O^\vartheta(T)}} = n_s/n_{O^\vartheta(T)}$. □

4. Quotient schemes and correspondence theorems

Let (X, S) be an association scheme, and T a closed subset of S. Recall that $X/T = \{xT \mid x \in X\}$, and $S//T = \{s^T \mid s \in S\}$. If $S//T$ is an association scheme on X/T, then $(X/T, S//T)$ is called the *quotient scheme of S over T*. If S is naturally valenced and T is a finite closed subset of S, then $S//T$ is an association scheme on X/T by [10, Theorem 4.1.3]. If S is naturally valenced and T is a (finite or infinite) normal closed subset of S, then $S//T$ is also an association scheme on X/T by [9, Theorem 1.5]. Schurian schemes provide examples that S may not be naturally valenced and the closed subset T may not be finite or normal but $S//T$ is an association scheme on X/T (see below for the details). In this section we will not assume that the closed subset T is finite or normal, but only assume that $S//T$ is an association scheme on X/T. Furthermore, we will not assume that S is naturally valenced either. So the known results about quotient schemes cannot be applied to the discussions in this section. The purpose of this section is to study the relationship between the closed subsets of S and the closed subsets of $S//T$. We will prove a correspondence theorem and generalize many known results. As an application, we will obtain the relationship between the closed subsets of the schurian scheme $S(A, G)$ and the subgroups of G containing A. This relationship will be needed in the next section.

Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be association schemes. A (*combinatorial*) *morphism* from (X, S) to $(\widetilde{X}, \widetilde{S})$ is a map $\phi : X \cup S \to \widetilde{X} \cup \widetilde{S}$ such that

(i) $\phi(X) \subseteq \widetilde{X}$ and $\phi(S) \subseteq \widetilde{S}$, and
(ii) for any $x, y \in X$ and $s \in S$ with $(x, y) \in s$, $(\phi(x), \phi(y)) \in \phi(s)$.

Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be association schemes. Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism. Then $\phi(1_X) = 1_{\widetilde{X}}$, and for any $s \in S$, $\phi(s^*) = \phi(s)^*$. The kernel of ϕ is defined by

$$\ker\phi := \{ s \in S \mid \phi(s) = 1_{\widetilde{X}} \}.$$

Then $\ker\phi$ is a closed subset of S. Furthermore, if ϕ is bijective, then we say that ϕ is an *isomorphism*, (X, S) and $(\widetilde{X}, \widetilde{S})$ are *isomorphic*, and denote $(X, S) \cong (\widetilde{X}, \widetilde{S})$. If ϕ is an isomorphism, then for any $p, q, r \in S$, the structure constants a_{pqr} of S and $a_{\phi(p)\phi(q)\phi(r)}$ of $\widetilde{S}$ are equal. In particular, if ϕ is an isomorphism, then for any $s \in S$, $n_s = n_{\phi(s)}$.

Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be association schemes, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism. ϕ is called a (*combinatorial*) *homomorphism* if for any $x, y \in X$ and any $s \in S$ with $(\phi(x), \phi(y)) \in \phi(s)$, there exist $u, v \in X$ such that $(u, v) \in s$, $\phi(u) = \phi(x)$, and $\phi(v) = \phi(y)$. Note that for a homomorphism $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$, the closed subset $\ker\phi$ may not be finite or normal. The next lemma says that for a surjective homomorphism $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$, the quotient $S/\!/\ker\phi$ is an association scheme on $X/\ker\phi$.

Lemma 4.1. *Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be association schemes, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a homomorphism. If ϕ is surjective, then $S/\!/\ker\phi$ is an association scheme on $X/\ker\phi$, and*

$$(X/\ker\phi, S/\!/\ker\phi) \cong (\widetilde{X}, \widetilde{S}).$$

Proof. Let $T = \ker\phi$. Then T is a closed subset of S. By [10, Lemma 5.1.4], for any $x, y \in X$, $xT = yT$ if and only if $\phi(x) = \phi(y)$, and for any $p, q \in S$ such that $p^T = q^T$, $\phi(p) = \phi(q)$. Thus, the map

$$\widetilde{\phi} : (X/T) \cup (S/\!/T) \to \widetilde{X} \cup \widetilde{S}, \quad xT \mapsto \phi(x),\ p^T \mapsto \phi(p), \quad \text{for any } x \in X,\ p \in S,$$

is well defined, and the restriction of $\widetilde{\phi}$ to X/T is injective. Let $p, q \in S$ such that $\phi(p) = \phi(q)$. Let $x, y \in X$ such that $(x, y) \in p$. Then $(\phi(x), \phi(y)) \in \phi(p)$. Hence, $(\phi(x), \phi(y)) \in \phi(q)$. But ϕ is a homomorphism. So there are $u, v \in X$ such that $(u, v) \in q$, $\phi(u) = \phi(x)$, and $\phi(v) = \phi(y)$. Thus, $uT = xT$ and $vT = yT$ by [10, Lemma 5.1.4]. But $(x, y) \in p$ and $(u, v) \in q$ imply that $(xT, yT) \in p^T$ and $(uT, vT) \in q^T$. Hence $p^T = q^T$. Therefore, the restriction of $\widetilde{\phi}$ to $S/\!/T$ is also injective. So $\widetilde{\phi}$ is injective. But ϕ is surjective. So $\widetilde{\phi}$ is also surjective, and hence $\widetilde{\phi}$ is bijective.

Let $x, y \in X$ and $p \in S$ such that $(xT, yT) \in p^T$. Then $(x, y) \in r$ for some $r \in TpT$. Hence $(\phi(x), \phi(y)) \in \phi(r)$. But $r^T = p^T$. So $\phi(r) = \phi(p)$ by [10, Lemma 5.1.4]. Thus, $(\phi(x), \phi(y)) \in \phi(p)$. That is, for any $xT, yT \in X/T$ and $p^T \in S/\!/T$ such that $(xT, yT) \in p^T$, $(\widetilde{\phi}(xT), \widetilde{\phi}(yT)) \in \widetilde{\phi}(p^T)$. Therefore, $(X/T, S/\!/T)$ is an association scheme, and $\widetilde{\phi}$ is an isomorphism. $\square$

Let (X, S) be an association scheme, and T a closed subset of S such that $(X/T, S/\!/T)$ is an association scheme. Then for any nonempty subset R of S, let $R/\!/T := \{r^T \mid r \in R\}$. The next proposition generalizes [10, Theorem 5.3.3], and has an interesting application to schurian schemes. For any nonempty subset P of S and any $x, y \in X$, if $(x, y) \in p$ for some $p \in P$, then we write $(x, y) \in P$.

Proposition 4.2. *Let (X, S) be an association scheme. Let T and R be closed subsets of S such that $R \supseteq T$ and both $(X/T, S/\!/T)$ and $(X/R, S/\!/R)$ are association schemes. Then $R/\!/T$ is a closed subset of $S/\!/T$, the quotient $(S/\!/T)/\!/(R/\!/T)$ is an association scheme on $(X/T)/(R/\!/T)$, and*

$$\big((X/T)/(R/\!/T), (S/\!/T)/\!/(R/\!/T)\big) \cong (X/R, S/\!/R).$$

Proof. It is clear that

$$\phi : (X/T, S//T) \to (X/R, S//R), \quad xT \mapsto xR,\ p^T \mapsto p^R, \quad \text{for any } xT \in X/T,\ p^T \in S//T,$$

is a surjective morphism, and $\ker\phi = R//T$. So $R//T$ is a closed subset of $S//T$. In the following we show that ϕ is a homomorphism. Let $xT, yT \in X/T$ and $p^T \in S//T$ such that $(\phi(xT), \phi(yT)) \in \phi(p^T)$. Then $(xR, yR) \in p^R$. Thus, $(x, y) \in RpR$. That is, $(x, y) \in r_1pr_2$ for some $r_1, r_2 \in R$. So there are $u, v \in X$ such that $(x, u) \in r_1$, $(u, v) \in p$, and $(v, y) \in r_2$. Hence, $(uT, vT) \in p^T$, $xR = uR$, and $yR = vR$. Thus, $\phi(uT) = \phi(xT)$, $\phi(vT) = \phi(yT)$, and ϕ is a homomorphism. Therefore, by Lemma 4.1, the quotient $(S//T)//(R//T)$ is an association scheme on $(X/T)/(R//T)$, and $((X/T)/(R//T), (S//T)//(R//T)) \cong (X/R, S//R)$. $\square$

The next two lemmas are needed in the proof of our main theorem in this section.

Lemma 4.3. *Let (X, S) be an association scheme, and let T be a closed subset of S such that $S//T$ is an association scheme on X/T. Then*

$$r^T \in p^Tq^T \quad \textit{if and only if} \quad r \in (TpT)(TqT), \quad \textit{for any } p, q, r \in S.$$

Proof. Let $p, q, r \in S$. If $r^T \in p^Tq^T$, then there are $x, y, z \in X$ such that $(yT, zT) \in r^T$, $(yT, xT) \in p^T$, and $(xT, zT) \in q^T$. Hence, $(y, z) \in r_1$ for some $r_1 \in TrT$, $(y, x) \in p_1$ for some $p_1 \in TpT$, and $(x, z) \in q_1$ for some $q_1 \in TqT$. Thus, $r_1 \in p_1q_1 \subseteq (TpT)(TqT)$. So $Tr_1T \subseteq (TpT)(TqT)$. But $TrT = Tr_1T$. Hence, $r \in (TpT)(TqT)$. On the other hand, if $r \in (TpT)(TqT)$, then $r \in p_1q_1$ for some $p_1 \in TpT$ and $q_1 \in TqT$. Hence, there are $x, y, z \in X$ such that $(y, z) \in r$, $(y, x) \in p_1$, and $(x, z) \in q_1$. So $(yT, zT) \in r^T$, $(yT, xT) \in p_1^T$, and $(xT, zT) \in q_1^T$. Thus, $r^T \in p_1^Tq_1^T$. But $p_1^T = p^T$, and $q_1^T = q^T$. Hence, $r^T \in p^Tq^T$. $\square$

Lemma 4.4. *Let (X, S) be an association scheme, and let T be a closed subset of S such that $S//T$ is an association scheme on X/T. Let R, R_1, and R_2 be nonempty subsets of S. Then the following hold.*

(i) *$R//T = TRT//T$.*
(ii) *$R_1//T \subseteq R_2//T$ if and only if $TR_1T \subseteq TR_2T$. In particular, $R_1//T = R_2//T$ if and only if $TR_1T = TR_2T$.*
(iii) *$(R_1//T)(R_2//T) = (R_1TR_2)//T$. In particular, for any positive integer n,*

$$(R//T)^n = (TRT)^n//T.$$

Proof. (i) Since $R \subseteq TRT$, we see that $R//T \subseteq TRT//T$. On the other hand, let $s \in TRT$. Then $s \in t_1rt_2$ for some $t_1, t_2 \in T$ and $r \in R$. Hence, $TsT \subseteq Tt_1rt_2T = TrT$. Thus, $TsT = TrT$, and $s^T = r^T$. So $TRT//T \subseteq R//T$. Therefore, (i) holds.

(ii) If $TR_1T \subseteq TR_2T$, then $R_1//T \subseteq R_2//T$ by (i). Now assume that $R_1//T \subseteq R_2//T$. Let $s \in TR_1T$. Then $s^T = r_1^T$ for some $r_1 \in R_1$ by (i). So $R_1//T \subseteq R_2//T$ implies that $s^T = r_2^T$ for some $r_2 \in R_2$. Thus, $TsT = Tr_2T$, and hence $s \in TR_2T$. This proves that $TR_1T \subseteq TR_2T$ if $R_1//T \subseteq R_2//T$. So (ii) holds.

(iii) Let $r_1 \in R_1$, $r_2 \in R_2$, and $s \in S$. Then by Lemma 4.3, $s^T \in r_1^Tr_2^T$ if and only if $s \in (Tr_1T)(Tr_2T)$. Thus, $(R_1//T)(R_2//T) = [(TR_1T)(TR_2T)]//T = (TR_1TR_2T)//T$. So $(R_1//T)(R_2//T) = (R_1TR_2)//T$ by (i), and (iii) holds. $\square$

Let (X, S) be an association scheme. Let P and Q be nonempty subsets of S. As in [10, Section 2.5], define $K_Q(P) := \{s \in Q \mid s^*Ps \subseteq P\}$. Note that if P is a closed subset of S, then P is strongly normal if and only if $K_S(P) = S$.

Now we are ready to prove the main result of this section.

Theorem 4.5 *(Correspondence Theorem). Let (X, S) be an association scheme, and let T be a closed subset of S such that $S//T$ is an association scheme on X/T. Let R be a nonempty subset of S. Then the following hold.*

(i) *$R//T$ is a closed subset of $S//T$ if and only if TRT is a closed subset of S.*
(ii) *Let V be a nonempty subset of S. Then $K_{V//T}(R//T) = K_V(TRT)//T$. In particular, $R//T$ is a strongly normal closed subset of $S//T$ if and only if TRT is a strongly normal closed subset of S.*
(iii) *$\langle R//T\rangle = \langle TRT\rangle//T$.*
(iv) *If R is a closed subset of S and $R \supseteq T$, then the thin residue $O^{\vartheta}(R//T) = \mathrm{snc}(T, R)//T$, where $\mathrm{snc}(T, R)$ is the strongly normal closure of T in R.*

Proof. (i) From Lemma 4.4,

$$R//T = (R//T)^*(R//T) \quad\Leftrightarrow\quad R//T = R^*TR//T$$
$$\Leftrightarrow\quad TRT = TR^*TRT = (TRT)^*(TRT).$$

So $R//T$ is a closed subset of $S//T$ if and only if TRT is a closed subset of S, and (i) holds.

(ii) Let $s \in S$. Then by Lemma 4.4,

$$\big(s^T\big)^*(R//T)s^T \subseteq R//T \quad\Leftrightarrow\quad \big(s^*TRTs\big)//T \subseteq R//T$$
$$\Leftrightarrow\quad Ts^*TRTsT \subseteq TRT$$
$$\Leftrightarrow\quad s^*(TRT)s \subseteq TRT.$$

Thus, for any $s \in V$, $s^T \in K_{V//T}(R//T)$ if and only if $s \in K_V(TRT)$. Hence, $K_{V//T}(R//T) = K_V(TRT)//T$. In particular, $K_{S//T}(R//T) = K_S(TRT)//T$, and $K_{S//T}(R//T) = S//T$ if and only if $K_S(TRT) = S$. So $R//T$ is a strongly normal closed subset of $S//T$ if and only if TRT is a strongly normal closed subset of S, and (ii) holds.

(iii) Since $\langle R//T\rangle = \langle (R \cup R^*)//T\rangle$ and $\langle TRT\rangle = \langle T(R \cup R^*)T\rangle$, without loss of generality, we may assume that $R = R^*$. So by (3.1) and Lemma 4.4,

$$\langle R//T\rangle = \bigcup_{n\in\mathbb{N}}(R//T)^n = \bigcup_{n\in\mathbb{N}}\big[(TRT)^n//T\big]$$
$$= \Big[\bigcup_{n\in\mathbb{N}}(TRT)^n\Big]//T = \langle TRT\rangle//T.$$

Hence, (iii) holds.

(iv) Note that if TRT is a closed subset of S, then $TRT = T(TRT) \supseteq T$. So (iv) follows from (ii). □

Theorem 4.5 generalizes many know results, and has interesting applications to schurian schemes. The next corollary is a direct consequence of Theorem 4.5.

Corollary 4.6. *(See [10, Lemmas 4.1.7, 4.2.2, and 4.2.4].) Let (X, S) be a naturally valenced association scheme, and T a finite closed subset of S. Let R be a nonempty subset of S. Then the following hold.*

(i) *If $TRT \subseteq R$, then $R//T$ is a closed subset of $S//T$ if and only if R is a closed subset of S.*
(ii) *If $R//T$ is a closed subset of $S//T$ and $U = \{s \in S \mid s^T \in R//T\}$, then U is a closed subset of S and $U//T = R//T$. (In fact, $U = TRT$.)*
(iii) *If $T \subseteq \langle R\rangle$, then $\langle R//T\rangle = \langle R\rangle//T$.*
(iv) *If U is a closed subset of S such that $T \subseteq U$ and $\langle R//T\rangle = U//T$, then $U = \langle R \cup T\rangle$.*
(v) *Let U be a closed subset of S such that $T \subseteq U$, and V a nonempty subset of S. Then $K_{V//T}(U//T) = K_V(U)//T$.*

Now we discuss applications of Theorem 4.5 to schurian schemes.

Let G be a group. For any $g \in G$, let $g^\tau := \{(a, b) \in G \times G \mid a^{-1}b = g\}$. Let $G^\tau := \{g^\tau \mid g \in G\}$. Then (G, G^τ) is a thin scheme. Furthermore, for any $g, h \in G$, $g^\tau h^\tau = \{(gh)^\tau\}$, and $(g^\tau)^* = (g^{-1})^\tau$. For more details about the thin scheme (G, G^τ), the reader is referred to [10, Section 5.5]. For a nonempty subset A of G, define $A^\tau := \{a^\tau \mid a \in A\}$. Then A^τ is a closed subset of G^τ if and only if A is a subgroup of G, and A^τ is a strongly normal closed subset of G^τ if and only if A is a normal subgroup of G.

Let G be a group, and A a nonempty subset of G. Then for any $x \in G$,

$$\begin{aligned} xA^\tau &= \{y \in G \mid (x,y) \in a^\tau \text{ for some } a^\tau \in A^\tau\} \\ &= \{y \in G \mid x^{-1}y = a \text{ for some } a \in A\} \\ &= xA. \end{aligned}$$

Assume that A is a subgroup of G. Then A^τ is a closed subset of G^τ, and for any $x \in G$, $xA^\tau = xA$ is a left coset of A in G. Thus, G/A^τ is the set of all left cosets of A in G. Furthermore, for any $x, y \in G$ and $g^\tau \in G^\tau$ such that $y \in xA^\tau g^\tau A^\tau$, since $A^\tau g^\tau A^\tau = (AgA)^\tau$, we see that $y \in xAgA$. So $y = xa_1ga_2$ for some $a_1, a_2 \in A$, and hence $(xA, yA) = (xA, xa_1ga_2A) = xa_1(A, gA) = u(A, gA)$, where $u = xa_1 \in G$. Therefore, for any $g^\tau \in G^\tau$,

$$\begin{aligned} (g^\tau)^{A^\tau} &= \{(xA^\tau, yA^\tau) \in (G/A^\tau) \times (G/A^\tau) \mid y \in xA^\tau g^\tau A^\tau\} \\ &= \{(xA, yA) \in (G/A^\tau) \times (G/A^\tau) \mid y \in xAgA\} \\ &= \{u(A, gA) \mid u \in G\} \in S(A, G). \end{aligned}$$

Thus, $G^\tau/\!/A^\tau = S(A, G)$, and the quotient $(G/A^\tau, G^\tau/\!/A^\tau)$ is a schurian scheme. Note that for any $g, h \in G$, $(g^\tau)^{A^\tau} = (h^\tau)^{A^\tau}$ if and only if $A^\tau g^\tau A^\tau = A^\tau h^\tau A^\tau$ if and only if $AgA = AhA$. However, if the subgroup A is not finite, then the closed subset A^τ is not finite, and hence (3.3) does not hold for $G^\tau/\!/A^\tau$.

Let G be a group, and H a nonempty subset of G. The *normal closure* of H in G, H^G, is the intersection of all normal subgroups of G that contain H. Clearly $H^G = \langle g^{-1}hg \mid g \in G, h \in H\rangle$.

From Proposition 4.2, Lemma 4.4, and Theorem 4.5, we have the following

Theorem 4.7. *Let G be a group, and A a subgroup of G. Let R, R_1, and R_2 be nonempty subsets of G. Then the following hold.*

(i) *$R_1^\tau/\!/A^\tau \subseteq R_2^\tau/\!/A^\tau$ if and only if $AR_1A \subseteq AR_2A$. In particular, $R_1^\tau/\!/A^\tau = R_2^\tau/\!/A^\tau$ if and only if $AR_1A = AR_2A$.*
(ii) $(R_1^\tau/\!/A^\tau)(R_2^\tau/\!/A^\tau) = (R_1AR_2)^\tau/\!/A^\tau$.
(iii) *$R^\tau/\!/A^\tau$ is a closed subset of $G^\tau/\!/A^\tau$ if and only if ARA is a subgroup of G.*
(iv) *$R^\tau/\!/A^\tau$ is a strongly normal closed subset of $G^\tau/\!/A^\tau$ if and only if ARA is a normal subgroup of G.*
(v) *If R is a subgroup of G and $R \supseteq A$, then $O^\vartheta(R^\tau/\!/A^\tau) = (A^R)^\tau/\!/A^\tau$, where A^R is the normal closure of A in R.*
(vi) *If R is a subgroup of G and $R \supseteq A$, then $R^\tau/\!/A^\tau$ is a closed subset of $G^\tau/\!/A^\tau$, $(G^\tau/\!/A^\tau)/\!/(R^\tau/\!/A^\tau)$ is an association scheme on $(G/A^\tau)/(R^\tau/\!/A^\tau)$, and*

$$\big((G/A^\tau)/(R^\tau/\!/A^\tau), (G^\tau/\!/A^\tau)/\!/(R^\tau/\!/A^\tau)\big) \cong (G/R^\tau, G^\tau/\!/R^\tau).$$

5. Applications to common divisor graphs of groups

In this section we study applications of IP-graphs of naturally valenced association schemes to common divisor graphs of groups. We will obtain the results in [6] as direct consequences of Theorems 3.8, 3.10, and 4.7.

Let G be a group, and A a subgroup of G. In this section we always assume that all subdegrees of (A, G) are finite and the common divisor graph of (A, G) has two non-trivial components with vertex sets D_1 and D_2, respectively, such that $\min(D \setminus \{1\}) \in D_1$, where D is the set of all subdegrees of (A, G). Thus, the schurian scheme $S(A, G)$ is naturally valenced, and the graph $\mathcal{IP}(S(A, G))$ has two non-trivial components. Recall that $S(A, G)$ and the quotient scheme $G^\tau/\!/A^\tau$ are the same, and for any $g \in G$, the valency of $(g^\tau)^{A^\tau}$ is $[AgA : A]$, the number of left cosets of A contained in AgA. Let

$$H := \bigcup_{[AgA:A] \in D_1 \cup \{1\}} AgA.$$

Note that for any $g \in N_G(A)$, $[AgA : A] = 1$. Hence, $N_G(A) \subset H \subset G$, where both inclusions are proper. Since

$$H^\tau // A^\tau = \{ (g^\tau)^{A^\tau} \in G^\tau // A^\tau \mid \text{the valency of } (g^\tau)^{A^\tau} \text{ is in } D_1 \cup \{1\} \},$$

from Lemma 3.6, $H^\tau // A^\tau$ is a closed subset of $G^\tau // A^\tau$ if and only if $\mathcal{IP}(S(A,G))$ is stable. But $H = AHA$. So $H^\tau // A^\tau$ is a closed subset of $G^\tau // A^\tau$ if and only if H is a subgroup of G by Theorem 4.7(iii). But by definition, $\mathcal{IP}(S(A,G))$ is stable if and only if A is stable in G. So H is a subgroup of G if and only if A is stable in G. Hence, we have the following corollary.

Corollary 5.1. *(See [6, Theorem A].) Let G be a group, and A a subgroup of G such that all subdegrees of (A,G) are finite and the common divisor graph of (A,G) has two non-trivial components. Let $H := \bigcup_{[AgA:A] \in D_1 \cup \{1\}} AgA$. Then H is a subgroup of G if and only if A is stable in G. In any case, $N_G(A) \subset H \subset G$, where both inclusions are proper.*

For the rest of this section we will always assume that A is stable in G. As above, let $H = \bigcup_{[AgA:A] \in D_1 \cup \{1\}} AgA$. Then H is a subgroup of G such that $H \supseteq A$, and $H^\tau // A^\tau$ is a closed subset of $G^\tau // A^\tau$. Let $v \in G \setminus H$. Then $(v^\tau)^{A^\tau} \in (G^\tau // A^\tau) \setminus (H^\tau // A^\tau)$ by Theorem 4.7(i). Hence, $O^\vartheta(H^\tau // A^\tau)(v^\tau)^{A^\tau} O^\vartheta(H^\tau // A^\tau) = \{(v^\tau)^{A^\tau}\}$ by Theorem 3.8(i). Let $L = A^H$, the normal closure of A in H. Then by Theorem 4.7(v), $O^\vartheta(H^\tau // A^\tau) = L^\tau // A^\tau$. Therefore, $(L^\tau // A^\tau)(v^\tau)^{A^\tau}(L^\tau // A^\tau) = \{(v^\tau)^{A^\tau}\}$. But L is a subgroup of H containing A. So $(L^\tau // A^\tau)(v^\tau)^{A^\tau}(L^\tau // A^\tau) = (LvL)^\tau // A^\tau$ by Theorem 4.7(ii). Thus, $(LvL)^\tau // A^\tau = \{(v^\tau)^{A^\tau}\}$, and hence $A^\tau v^\tau A^\tau = (LvL)^\tau$ by Theorem 4.7(i). So $AvA = LvL$. Furthermore, since the index $|L : A|$ is equal to the valency of the set $L^\tau // A^\tau$, and $L^\tau // A^\tau = O^\vartheta(H^\tau // A^\tau)$, we see that $|L : A|$ is equal to the valency of the set $O^\vartheta(H^\tau // A^\tau)$. Thus, $|L : A|$ is finite and divides the valency of $(v^\tau)^{A^\tau}$ for any $v \in G \setminus H$ by Theorem 3.8(i). That is, $|L : A|$ divides all subdegrees in D_2. Let M be a subgroup of G containing A. Then $M^\tau // A^\tau$ is a closed subset of $G^\tau // A^\tau$ by Theorem 4.7(iii). Thus, we have either $M^\tau // A^\tau \subseteq H^\tau // A^\tau$ or $M^\tau // A^\tau \supseteq O^\vartheta(H^\tau // A^\tau)$ by Theorem 3.8(ii). Hence by Theorem 4.7(i), we have either $M \subseteq H$ or $M \supseteq L$. So we have the following corollary.

Corollary 5.2. *(See [6, Theorem B].) Let G be a group, and A a subgroup of G. Assume that all subdegrees of (A,G) are finite, A is stable in G, and the common divisor graph of (A,G) has two non-trivial components. Let $H = \bigcup_{[AgA:A] \in D_1 \cup \{1\}} AgA$, and $L = A^H$. Then the following hold.*

- (i) *$AvA = LvL$ for every $v \in G \setminus H$.*
- (ii) *The index $|L : A|$ is finite and divides the greatest common divisor of subdegrees in D_2.*
- (iii) *If M is a subgroup of G containing A, then either $M \subseteq H$ or $M \supseteq L$.*

Since $H = \bigcup_{[AgA:A] \in D_1 \cup \{1\}} AgA$ is a subgroup of G containing A, the set of subdegrees of (A,H) is $D_1 \cup \{1\}$. Since $L = A^H$ and $O^\vartheta(H^\tau // A^\tau) = L^\tau // A^\tau$, the quotient scheme $(G^\tau // A^\tau) // O^\vartheta(H^\tau // A^\tau)$ is isomorphic to the schurian scheme $G^\tau // L^\tau$ by Theorem 4.7(vi). So the set of valencies of elements in $G^\tau // L^\tau$ and the set of valencies of elements in $(G^\tau // A^\tau) // O^\vartheta(H^\tau // A^\tau)$ are the same. Thus, by Theorem 3.10, the set of subdegrees of (L,G) is $\{m/|L : A| \mid m \in D_2\} \cup \{1\}$. Therefore, we have the following corollary.

Corollary 5.3. *(See [6, Theorem C].) Let G be a group, and A a subgroup of G. Assume that all subdegrees of (A,G) are finite, A is stable in G, and the common divisor graph of (A,G) has two non-trivial components. Let $H = \bigcup_{[AgA:A] \in D_1 \cup \{1\}} AgA$, and $L = A^H$. Then the set of subdegrees of (A,H) is $D_1 \cup \{1\}$, and the set of subdegrees of (L,G) is $\{m/|L : A| \mid m \in D_2\} \cup \{1\}$.*

The next corollary is a direct consequence of Corollary 3.9.

Corollary 5.4. *Let G be a group, and A a subgroup of G. Assume that all subdegrees of (A, G) are finite, A is stable in G, and the common divisor graph of (A, G) has two non-trivial components. Let $H = \bigcup_{[AgA:A]\in D_1\cup\{1\}} AgA$, and $v \in G \setminus H$. Then for any $u \in HvH$, $[AuA : A] = [AvA : A]$.*

Acknowledgments

The author would like to thank the referees for their useful comments. The author would also like to thank P.-H. Zieschang for discussions on the Correspondence Theorem.

References

[1] E.A. Bertram, M. Herzog, A. Mann, On a graph related to conjugate classes of groups, Bull. London Math. Soc. 22 (1990) 569–573.
[2] G.M. Bergman, H.W. Lenstra Jr., Subgroups close to normal subgroups, J. Algebra 127 (1989) 80–97.
[3] R. Camina, Schemes and the IP-graph, J. Algebraic Combin. 28 (2008) 271–279.
[4] P. Cameron, Permutation Groups, London Math. Soc. Stud. Texts, vol. 45, Cambridge University Press, Cambridge, 1999.
[5] I.M. Isaacs, C.E. Praeger, Permutation group subdegrees and the common divisor graph, J. Algebra 159 (1993) 158–175.
[6] G. Kaplan, On groups admitting a disconnected common divisor graph, J. Algebra 193 (1997) 616–628.
[7] P.M. Neumann, Coprime suborbits of transitive permutation groups, J. London Math. Soc. (2) 47 (2) (1993) 285–293.
[8] B. Xu, Valencies and IP-graphs of association schemes, J. Algebra 321 (2009) 2521–2539.
[9] B. Xu, Morphisms of association schemes and quotient scheme, in preparation, 2009.
[10] P.-H. Zieschang, Theory of Association Schemes, Springer Monogr. Math., Springer-Verlag, Berlin, 2005.